Amplitude Dependence of Time of Flight

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Abstract. Machida found in tracking studies [Shinji Machida, presentation at the FFAG05 Workshop, Kyoto University Research Reactor Institute, Osaka, Japan, 5–9 December 2005] that the time of flight in a linear non-scaling FFAG depended on the transverse amplitude of the particles. I compute a relationship between the transverse amplitude dependence of the time of flight and the variation of tune with energy and explain its physical origin.

Keywords: Fixed Field Alternating Gradient Accelerator; Simulation; Tracking
PACS: 29.27.Bd

When doing tracking studies for linear non-scaling FFAGs, Machida [1] found that particles with large enough transverse amplitude did not get accelerated in a linear non-scaling FFAG. He found that this was likely caused by a dependence of the time of flight in the FFAG on the transverse amplitude (Fig. 1).

It turns out that there is necessarily a relationship between the variation of the time of flight with transverse amplitude and the variation of the tune with energy. This relationship can be straightforwardly computed. First of all, one finds the closed orbit $\left(x_0(s,E), p_{x0}(s,E), y_0(s,E), p_{y0}(s,E)\right)$ in the machine without RF as a function of the energy E, and the linear map about the closed orbit. One can then make a change of phase space variables into "normalized" variables, where the Hamiltonain can be written as

$$H = H_0(s,E) + \frac{\partial \psi_x(s,E)}{\partial s} J_x + \frac{\partial \psi_y(s,E)}{\partial s} J_y + O(J_x^{3/2}) + O(J_y^{3/2}), \tag{1}$$

where J_x and J_y are defined for an uncoupled system as usual as

$$J_x = \frac{1}{2}\left(\gamma_x(s,E)\bar{x}^2 + 2\alpha_x(s,E)\bar{x}\bar{p}_x + \beta_x(s,E)\bar{p}_x^2\right) \qquad J_y = \frac{1}{2}\left(\gamma_y(s,E)\bar{y}^2 + 2\alpha_y(s,E)\bar{y}\bar{p}_y + \beta_y(s,E)\bar{p}_y^2\right) \tag{2}$$

$$x = \bar{x} + x_0(s,E) \qquad y = \bar{y} + y_0(s,E) \qquad p_x = \bar{p}_x + p_{x0}(s,E) \qquad p_y = \bar{p}_y + p_{y0}(s,E). \tag{3}$$

Here $\beta_{x,y}(s,E)$, $\alpha_{x,y}(s,E)$, and $\gamma_{x,y}(s,E)$ are the Courant-Snyder functions, and $\psi_{x,y}(s,E)$ is the local phase advance, which is related to the tune for a system with period L by $\psi_{x,y}(s+L,E) - \psi_{x,y}(s,E) = 2\pi\nu(E)$. Other than $\psi_{x,y}(s,E)$, all of the above functions of s are periodic in s with period L. Note that the usual scaling of the Hamiltonian and the transverse momenta by the total momentum $p = \sqrt{E^2/c^2 - (mc)^2}$ has occurred (m is the particle mass, c is the speed of light). In the more general case with coupling, one still transforms the Hamiltonain to (1), but now the transformation can be written more generally as $\bar{\mathbf{z}} = A(s,E)\mathbf{z}_N$, with $\bar{\mathbf{z}} = (\bar{x}, \bar{p}_x, \bar{y}, \bar{p}_y)$ and $\mathbf{z}_N = (\sqrt{2J_x}\cos\theta_x, -\sqrt{2J_x}\sin\theta_x, \sqrt{2J_y}\cos\theta_y, -\sqrt{2J_y}\sin\theta_y)$. As long as the motion is stable at the energy E, it is always possible to do this transformation. As part of this transformation, the time of flight t was transformed to a new time of flight t_N:

$$t = t_N + p\frac{\partial p_{x0}}{\partial E}\bar{x} - p\frac{\partial x_0}{\partial E}\bar{p}_x + p\frac{\partial p_{y0}}{\partial E}\bar{y} - p\frac{\partial y_0}{\partial E}\bar{p}_y + \frac{p}{2}\mathbf{z}_N^T A^T(s,E) S \frac{\partial A(s,E)}{\partial E}\mathbf{z}_N. \tag{4}$$

Note that t differs from t_N only by a periodic function of s. Here S is the matrix of the symplectic metric,

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{5}$$

Now, consider the the quantity $T = t(s+L) - t(s)$, the time of flight through one period of the lattice. For linear oscillations about the energy-dependent closed orbit, this is just $t_N(s+L) - t_N(s)$ plus terms that are oscillatory

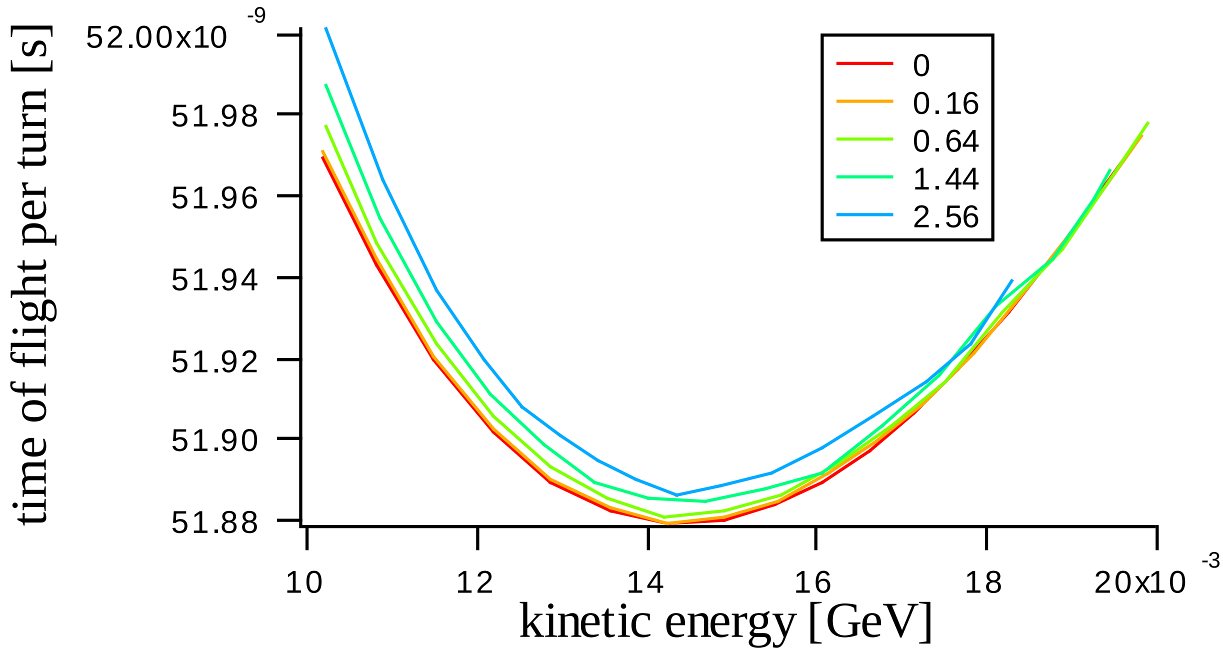

FIGURE 1. Relationship between time of flight and transverse amplitude [1]. The legend indicates an initial transverse displacement for the corresponding time of flight curves. Results were based on tracking and averaging out the oscillatory pieces.

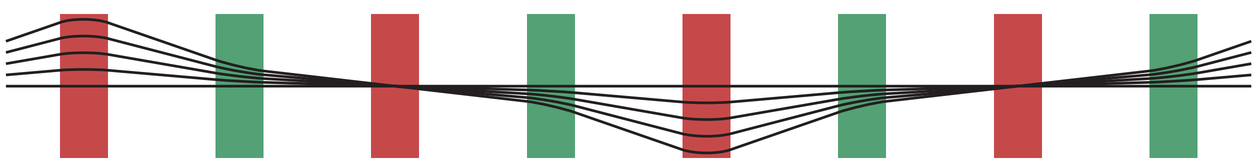

FIGURE 2. Betatron oscillations, showing longer path lengths for larger amplitude oscillations.

functions of the angle variables θ_x and θ_y. We can compute $t_N(s+L) - t_N(s)$ from Hamilton's equations of motion:

$$t_N(s+L) - t_N(s) = -p\int_0^L \frac{\partial H}{\partial E} ds = -p\int_s^{s+L} \frac{\partial H_0(s,E)}{\partial E} ds - 2\pi p \frac{d\nu_x}{dE} J_x - 2\pi p \frac{d\nu_y}{dE} J_y + O(J_x^{3/2}) + O(J_y^{3/2}). \quad (6)$$

The first term on the right hand side is the time of flight for a period along the closed orbit. The next two terms are the interesting result; they show a relationship between the time-of-flight dependence on transverse amplitude and the tune variation with energy.

What is the physical origin of this behavior? First, think about a straight beam line. For finite transverse oscillations, the beam makes an angle with the axis (Fig. 2), and thus the path length (and therefore the time of flight) along the particle trajectory increases as the square of the angle (and therefore linearly in $J_{x,y}$). The tune decreases with increasing energy (for large enough energy), while the path length increases with increasing transverse amplitude.

What about the result that correcting the chromaticity eliminates the time of flight variation with amplitude? Consider a lattice with chromaticity correcting sextupoles. Assume a lattice with positive dispersion and a phase advance per cell of less than π. To make the horizontal chromaticity more positive, the gradient should become more positive with increasing x. This means that the vertical field from the sextupoles should be a positive parabola in the midplane. Now, consider horizontal betatron oscillations: the average field from the sextupole will not be zero, but will be positive and proportional to the square of the maximum orbit displacement (in other words, proportional to J_x). This additional bending field will tend to reduce the orbit radius, decreasing the path length, and therefore decreasing the time of flight. Similarly, to make the vertical chromaticity more positive, the vertical gradient of the horizontal field should become more negative with increasing x. From the divergence equation in Maxwell's equations, this means that the vertical field should be a positive parabola in y. Again, that means that averaging over vertical betatron oscillations, the average vertical field is positive and proportional to J_y, displacing the beam to a smaller radius as with the horizontal oscillations.

These physical explanations are not meant to be quantitative; Eq. (6) gives the quantitative result. The time of flight dependence on transverse amplitude can thus be computed straightforwardly from the tunes for oscillations about the closed orbit as a function of energy. These tunes are generally calculated early in the design process for an FFAG, and so the importance of the time of flight dependence on transverse amplitude can be quickly computed.

We determined that the transverse amplitude dependence of the time of flight shown in Fig. 1 is correctly predicted by Eq. (6).

ACKNOWLEDGMENTS

This work has been supported by the United States Department of Energy, Contract No. DE-AC02-98CH10886.

REFERENCES

1. Shinji Machida, presentation at the FFAG05 Workshop, Kyoto University Research Reactor Institute, Osaka, Japan, 5–9 December 2005.